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APPROXIMATIONS ON ANGULAR DISTRIBUTION OF INTENSITY OF THERMAL RADIATION

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Abstract—In engineering literature, the approximation $E_2(t) \simeq a\,e^{-bt}$ is frequently used and the values of a and b are obtained by curve fitting. The values of a and b are shown to be related to the assumed angular distribution of intensity by using Eddington's approximation. It is formally shown that $E_2(t) \simeq 0{\cdot}348 \exp[-1{\cdot}1613t] + 0{\cdot}652 \exp[-2{\cdot}941t]$ is equivalent to retaining the first four terms in the spherical harmonics expansion of intensity and the generalization of this procedure is indicated.

NOMENCLATURE

a_i, b_i, constants in equation (19);
A_n, coefficients of Legendre polynomials;
B, $= (\sigma/\pi)\,T^4$: integrated Planck's function;
c, velocity of light;
I, intensity of thermal radiation;
k, absorption coefficient;
p^R, pressure of radiation;
P_n, Legendre polynomials;
q^R, radiative energy flux in the positive x-direction;
u^R, radiative energy density;
τ, optical thickness.

1. INTRODUCTION

IN ONE-DIMENSIONAL radiation gas dynamics (RGD), the intensity of thermal radiation depends on position which is expressed non-dimensionally as optical thickness and the angle which the ray makes with the preferred direction. The energy flux, the energy density, and pressure of radiation are obtained as moments of intensity and are given by integral expressions involving the temperature of the whole flow field in the quasi-equilibrium transfer theory [1, 2]. The latter two quantities are usually negligible compared with the corresponding mechanical quantities and the influence of radiation is felt through the flux of energy. The energy equation of RGD becomes an integro-differential equation due to the radiation of flux term.

The mathematical difficulties of working with the integral expressions have prompted a number of approximations which can be broadly classified as (a) approximations based on the optical thickness of the medium and (b) approximations on the angular distribution of intensity. Approximations of the first category, in general, simplify the flux term by reducing the integral expression to an expression depending on local properties only. The examples are the two asymptotic optically thin and optically thick cases [3]. The optical thickness, by definition, involves the absorption coefficient and the characteristic dimension of the medium so that asymptotic cases may arise in a number of physical situations but a wide range of optical thicknesses lies beyond their scope. In contrast, approximations based on angular distribution of intensity are not restricted by the optical thickness of the medium and they are discussed, for example, in the references [4–9].

In the one-dimensional flux expression, the angular dependence of intensity gives rise to the integro-exponential function $E_2(t)$ which may be written as a sum of exponential functions

$$\Sigma\, a_i \exp[-b_i t].$$

Physically, this amounts to replacing the continuous angular dependence by discrete directional dependence. The first term in the sum is

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usually used for simplicity, as an *approximation*, that is, $E_2(t) \simeq a \exp[-bt]$ and the values of the constants a and b are found by curve fitting and by asymptotic considerations [10]. In this work, the value of the constants a and b is shown to be related to the assumed angular distribution of intensity by using Eddington's approximation and by considering the differential equation satisfied by the flux of energy. In a formal manner, it is shown that the expansion

$$E_2(t) = \Sigma\, a_i \exp[-b_i t]$$

is equivalent to the spherical harmonics method. In particular, the first two terms in the expansion of $E_2(t)$ and the first four terms in spherical harmonics expansion lead to the same fourth order differential equation for flux and by comparison,

$$E_2(t) \simeq 0{\cdot}348 \exp[-1613t] + 0{\cdot}652 \exp[-2{\cdot}941t]$$

is obtained. The procedure used is easily generalized.

2. EDDINGTON'S APPROXIMATION

Assuming local thermodynamic equilibrium, the equation of radiative transfer for a gray gas is given by

$$\mu \frac{dI}{d\tau} = B - I \tag{1}$$

where $\mu = \cos\theta$ and τ is the optical thickness, $d\tau = k\, dx$. Figure 1 shows the geometry for one-dimensional flux with the boundaries separated by an absorbing and emitting medium. If equation (1) is multiplied by $d\omega\ (= 2\pi\, d\mu)$ and $\mu\, d\omega$ and integrated over all the solid angles, then the following exact relations are obtained.

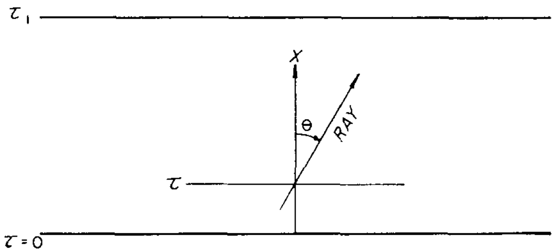

FIG. 1. One-dimensional radiative flux.

$$\frac{dq^R}{d\tau} = 4\pi B - cu^R \tag{2}$$

$$c\frac{dp^R}{d\tau} = -q^R \tag{3}$$

Geometrically, Eddington's approximation consists of the following:

$$I(\tau, \theta) = I_1 \quad \text{for } 0 < \theta < \pi/2 \tag{4}$$

$$I(\tau, \theta) = I_2 \quad \text{for } \pi/2 < \theta < \pi \tag{5}$$

Here I_1 and I_2 are functions of τ but independent of θ. Using the relations (4) and (5), we can evaluate u^R, q^R, and p^R as follows:

$$\left.\begin{aligned} u^R &= \frac{1}{c}\int_{4\pi} I\, d\omega = \frac{2\pi}{c}(I_1 + I_2) \\ q^R &= \int_{4\pi} \mu I\, d\omega = \pi(I_1 - I_2) \\ p^R &= \frac{1}{c}\int_{4\pi} \mu^2 I\, d\omega = \frac{2\pi}{c}(I_1 + I_2) = \frac{u^R}{3} \end{aligned}\right\} \tag{6}$$

Equation (3) can now be written as

$$\frac{c}{3}\frac{du^R}{d\tau} = -q^R \tag{7}$$

and combining this with equation (2), the flux q^R satisfies

$$\frac{d^2q^R}{d\tau^2} = 3q^R + 4\pi\frac{dB}{d\tau} \tag{8}$$

Differential equations for u^R and p^R can be obtained similarly from equations (2) and (7).

The same form of the differential equation is obtained if the flux is formulated as an integral and the approximation $E_2(t) \simeq a \exp[-bt]$ is used. That is

$$q^R \simeq \int_0^\tau 2\pi B\{a \exp[-b(\tau - t)]\}\, dt - \int_\tau^{\tau_1} 2\pi B\{a \exp[-b(t - \tau)]\}\, dt$$

In writing the expression for flux transparent boundaries are assumed, but the results can be shown to be valid for radiating boundaries. By twice differentiating with respect to τ, we get

$$\frac{\mathrm{d}^2 q^R}{\mathrm{d}\tau^2} = b^2 q^R + 4\pi a \frac{\mathrm{d}B}{\mathrm{d}\tau} \tag{9}$$

Comparison of equations (8) and (9) shows that the particular values $a = 1$ and $b = \sqrt{3}$ correspond to Eddington's approximation as far as flux is concerned. It is also seen that only these values of a and b provide the correct optically thin limit, namely $(\mathrm{d}g^R/\mathrm{d}\tau) = 4\pi B$ and the optically thick limit, $q^R = (4\pi/3)(\mathrm{d}B/\mathrm{d}\tau)$ for flux. A comparison of $E_2(t) \simeq \exp[-3^{\frac{1}{2}}t]$ with the exact curve in Fig. 2 reveals the above qualitative features.

There are other ways of obtaining Eddington's approximation, for example, the spherical harmonics method. As noted by Milne [1, p. 121], expansion of intensity in odd powers of $\cos\theta$ alone will also lead to $p^R = u^R/3$. However, the simple physical meaning is that isotropic radiation with different magnitudes of intensity in the positive and negative x-direction is incident on an area normal to the x-axis. The implicit assumption is that the angular distribution remains the same for all optical thicknesses. With this in mind, one can, in general, separate the variables τ and θ:

$$\begin{aligned} I(\tau, \theta) &= I_1(\tau)\, F(\mu) \qquad && 0 < \theta < \pi/2 \\ &= I_2(\tau)\, F(\mu) \qquad && \pi/2 < \theta < \pi \end{aligned} \tag{10}$$

Physically, this means that the angular distribution of opposite fluxes is a curve of a general type and for $F(\mu) = 1$, Eddington's approximation is obtained. From the relations (10), we get that u^R and p^R are related as

$$p^R = \frac{\int_0^1 \mu^2 F(\mu)\, \mathrm{d}\mu}{\int_1^0 F(\mu)\, \mathrm{d}\mu}\, u^R$$

That is, for any distribution $F(\mu)$, we have

$$b^{-2} = \frac{\int_0^1 \mu^2 F(\mu)\, \mathrm{d}\mu}{\int_0^1 F(\mu)\, \mathrm{d}\mu} \tag{11}$$

The relation (11) shows that various values of b suggested curve fitting to $E_2(t)$ correspond to approximating the angular distribution suitably by a proper choice of $F(\mu)$. Then the value of a may be obtained by asymptotic considerations; namely, $a = 1$ for the correct optically thin limit and $a = b^2/3$ for the optically thick limit. We may also note that proper boundary conditions can be extracted once $F(\mu)$ is chosen to suit the situation. This is a definite advantage over simple curve fitting.

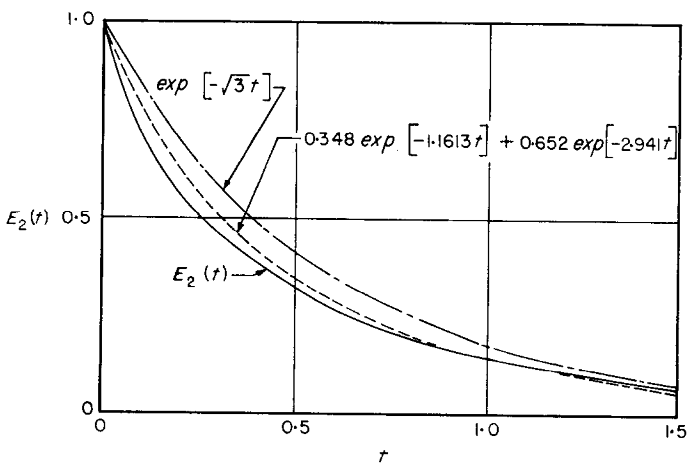

FIG. 2. The exact function and exponential approximations of $E_2(t)$.

3. EQUIVALENCE OF SPHERICAL HARMONICS METHOD AND EXPANSION OF INTEGRO-EXPONENTIAL FUNCTION $E_2(t)$

The intensity of radiation can be expanded in terms of Legendre polynomials to take into account the general variation of intensity with optical thickness and the angle θ. For practical purposes, the series has to be terminated at some stage and the termination results in additional relations. For example, let [12]

$$I(\tau, \mu) = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n + 1) A_n(\tau) P_n(\mu) \quad (12)$$

Using the orthogonality relation, we get

$$A_n = \int_{4\pi} I(\tau, \mu) P_n(\mu) \, d\omega \quad (13)$$

Multiplying equation (1) by $(2n + 1) P_n(\mu)$, using the recurrence relation and integrating over all the solid angles, we get

$$(n + 1) \frac{dA_{n+1}}{d\tau} + n \frac{dA_{n-1}}{d\tau} = -(2n + 1) A_n + 4\pi B \delta_{0n} \quad (14)$$

where $\delta_{0n} = 1$ if $n = 0$ and 0 otherwise.

Only the first three A's have physical meaning and from equation (13) they are

$$\left.\begin{aligned} A_0(\tau) &= \int_{4\pi} I(\tau, \mu) \, d\omega = cu^R \\ A_1(\tau) &= \int_{4\pi} \mu I(\tau, \mu) \, d\omega = q^R \\ A_2(\tau) &= \int_{4\pi} I(\tau, \mu)[\tfrac{3}{2}\mu^2 - \tfrac{1}{2}] \, d\omega = \tfrac{3}{2} cp^R \\ &\quad - \tfrac{1}{2} cu^R \end{aligned}\right\} \quad (15)$$

Writing the first two of the infinite set of equations (14):

$$n = 0: \quad \frac{dA_1}{d\tau} = -A_0 + 4\pi B \quad (16)$$

$$n = 1: \quad 2\frac{dA_2}{d\tau} + \frac{dA_0}{d\tau} = -3A_1 \quad (17)$$

Termination of the series after the first two terms amounts to setting $A_2 = 0$. From the last relation of (15), and combining equations (16) and (17), we get

$$u^R = 3p^R \quad \text{and} \quad \frac{d^2 A_1}{d\tau^2} = 3A_1 + 4\pi \frac{dB}{d\tau}$$

These are the same as those obtained by Eddington's approximation and have the physical meaning discussed previously. For higher approximations by retaining more terms in equation (12), the physical meaning gets obscure but higher order differential equations for flux are obtained.

Terminating the series (12) after the fourth term by setting $A_4 = 0$ and eliminating A_2 and A_3 between the equations obtained by $n = 2$ and $n = 3$, we get

$$\frac{d^4 A_1}{d\tau^4} - 10 \frac{d^2 A_1}{d\tau^2} + \frac{35}{3} A_1 - \frac{d^3}{d\tau^3}(4\pi B) + \frac{35}{9} \frac{d}{d\tau}(4\pi B) = 0 \quad (18)$$

This equation was obtained by Traugott [11] by the moment method suggested by Krook [9].

An equation of the same type as (18) can be obtained in a formal way by writing

$$E_2(\tau) = \sum_{i=1}^{\infty} a_i \exp[-b_i \tau] \quad (19)$$

and retaining the first two terms.

By successive differentiation of q_R with respect to τ, and using equation (19), we get:

$$\left.\begin{aligned} \frac{d^4}{d\tau^4}\left(\frac{q^R}{2\pi}\right) &= \int_0^{\tau} B\{a_1 b_1^4 \exp[-b_1(\tau - t)] \\ &\quad + a_2 b_2^4 \exp[-b_2(\tau - t)]\} \, dt \\ &\quad - \int_{\tau}^{\tau_1} B\{a_1 b_1^4 \exp[-b_1(t - \tau)] \\ &\quad + a_2 b_2^4 \exp[-b_2(t - \tau)]\} \, dt \\ &\quad + 2(a_1 + a_2) \frac{d^3 B}{d\tau^3} + 2(a_1 b_1^2 \\ &\quad + a_2 b_2^2) \frac{dB}{d\tau} \end{aligned}\right\} \quad (20)$$

Equation (20) can be simplified as follows:

$$\left.\begin{aligned} &a_1 b_1^4 \exp[-b_1(|t-\tau|)] \\ &\quad + a_2 b_2^4 \exp[-b_2(|t-\tau|)] \\ &\quad \equiv P\{a_1 b_1^2 \exp[-b_1(|t-\tau|)] \\ &\quad + a_2 b_2^2 \exp[-b_2(|t-\tau|)]\} \\ &\quad + Q\{a_1 \exp[-b_1(|t-\tau|)] \\ &\quad + a_2 \exp[-b_2(|t-\tau|)]\} \end{aligned}\right\} \quad (21)$$

Here P and Q are constants and are evaluated by equating the coefficients of

$$a_1 \exp[-b_1(|t-\tau|)]$$

and

$$a_2 \exp[-b_2(|t-\tau|)].$$

The result is

$$P = b_1^2 + b_2^2; \quad Q = -\, b_1^2 b_2^2 \qquad (22)$$

Now the differential equation satisfied by the flux is

$$\left.\begin{aligned} &\frac{d^4 q^R}{d\tau^4} - (b_1^2 + b_2^2)\frac{d^2 q^R}{d\tau^2} + b_1^2 b_2^2 q^R \\ &\quad - (a_1 + a_2)\frac{d^3}{d\tau^3}(4\pi B) + [(a_1 + a_2) \\ &\quad (b_1^2 + b_2^2) - (a_1 b_1^2 + a_2 b_2^2)] \\ &\quad \frac{d}{d\tau}(4\pi B) = 0 \end{aligned}\right\} \quad (23)$$

Comparing this equation with equation (18), they are seen to be of the same form and the constants a_1, a_2, b_2 can be evaluated. The result is

$$\left.\begin{aligned} &a_1 = 0{\cdot}348 \quad a_2 = 0{\cdot}652 \quad b_1^2 = 1{\cdot}35 \\ &\qquad\qquad b_2^2 = 8{\cdot}65 \\ &E_2(\tau) \simeq 0{\cdot}348 \exp[-1{\cdot}1613\tau] \\ &\qquad\qquad + 0{\cdot}652 \exp[-2{\cdot}941\tau] \end{aligned}\right\} \quad (24)$$

Traugott [11] evaluated E_3 and E_4, using a gas slab with a linear distribution of $B(t)$. His results are

$$\left.\begin{aligned} &E_3(\tau) = 0{\cdot}2645 \exp[-1{\cdot}1612\tau] \\ &\qquad\qquad + 0{\cdot}2355 \exp[-2{\cdot}942\tau] \\ &E_4(\tau) = 0{\cdot}27 \exp[-1{\cdot}1612\tau] \\ &\qquad\qquad + 0{\cdot}0634 \exp[-2{\cdot}942\tau] \end{aligned}\right\} \quad (25)$$

In equations (24) and (25), it is seen that the exponents have the same value. Integration of $E_2(t)$ in equation (24) gives E_3 and E_4, but the numerical coefficients differ from those of equation (25). Equation (24) is plotted in Fig. 2 and it is seen that two terms represent $E_2(t)$ much better than one.

The procedure of this section is easily generalized. By successive differentiation of equation (19) and depending on the number of terms retained, the algebraic procedure of equations (21) and (22) will lead to a differential equation for radiative flux.

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Résumé—Dans la littérature technique, l'approximation $E_2(t) \simeq a\,\mathrm{e}^{-bt}$ est fréquemment utilisée et les valeurs de a et b sont obtenues par ajustage d'une courbe. On montre que les valeurs de a et b sont reliées à la distribution angulaire d'intensité hypothétique en utilisant l'approximation d'Eddington. On montre analytiquement que l'expression $E_2(t) \simeq 0{,}348 \exp[-1{,}1613t] + 0{,}652 \exp[-2{,}941t]$ revient à retenir les quatre premiers termes du développement de l'intensité en harmoniques sphériques et l'on indique la généralisation de ce procédé.

Zusammenfassung—In der technischen Literatur wird häufig die Näherung $E_2(t) \simeq a\,\mathrm{e}^{-bt}$ verwendet und man erhält die Werte a und b durch Kurvenanpassung. Durch Anwendung von Eddington's Näherung wird die Beziehung der Werte a und b mit der angenommenen Winkelverteilung der Strahlungsdichte angegeben. Es wird gezeigt, dass der Ausdruck $E_2(t) \simeq 0{,}348 \exp[-1{,}1613t] + 0{,}652 \exp[-2{,}941t]$ der Beibehaltung der ersten vier Terme bei der kugelförmigen, harmonischen Ausbreitung der Strahlungsdichte gleichwertig ist; auf die Verallgemeinerung dieses Verfahrens wird hingewiesen.

Аннотация—В технической литературе часто используется аппроксимация $E_2(t) \simeq a\,\mathrm{e}^{-bt}$, а значения a и b получают графически. С помощью аппроксимации Эддингтона выведена связь между значениями a и b и предполагаемым угловым распределение интенсивности.

Показано, что $E_2(t) \simeq 0{,}348 \exp[-1{,}1613t] + 0{,}652 \exp[-2{,}941t]$ равно первым четырем сохраняющимся членам в разложении интенсивности по шаровым гармоническим функциям, и приводится обобщение этого приема.